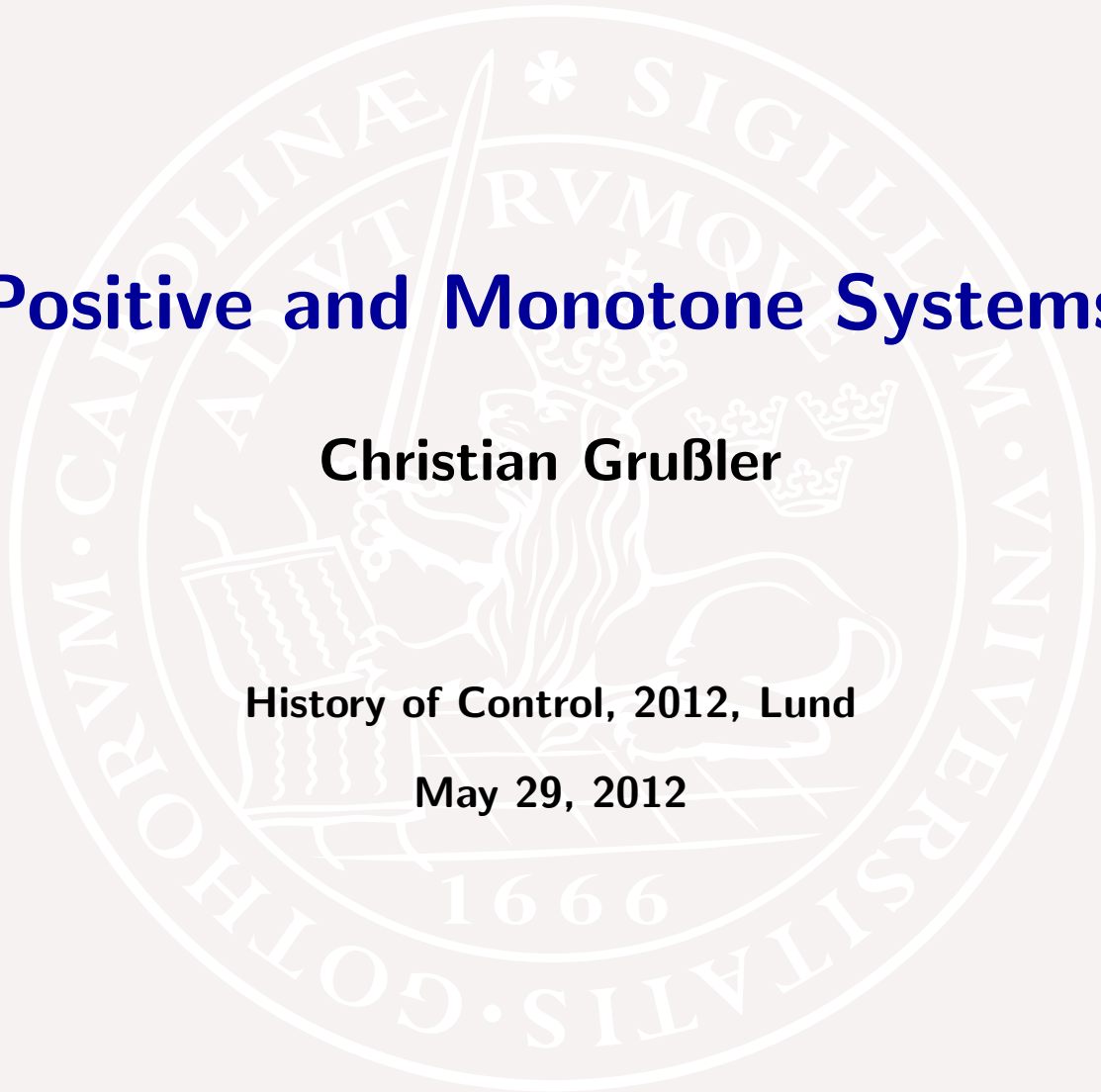


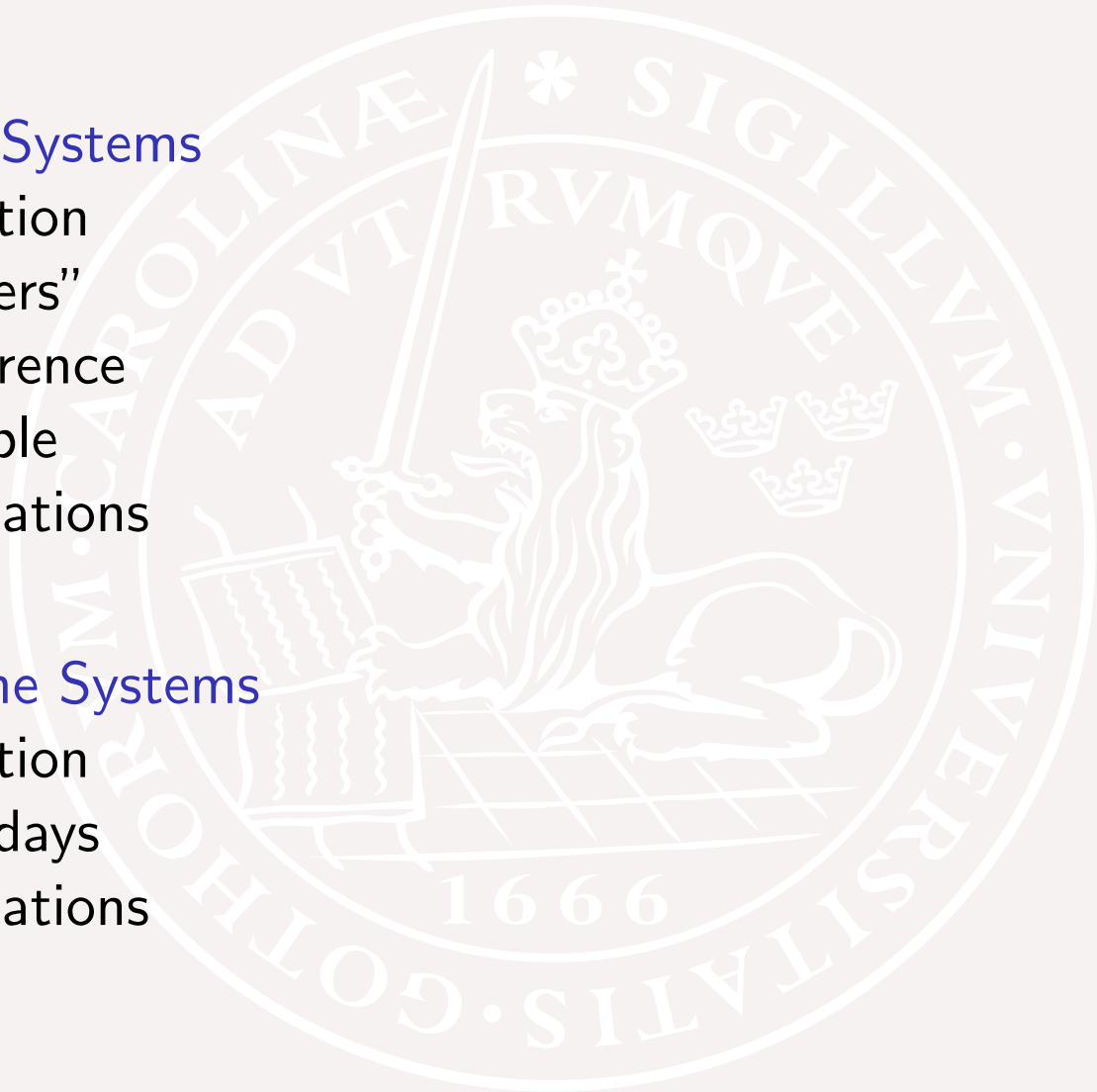
Positive and Monotone Systems

Christian Grußler

History of Control, 2012, Lund

May 29, 2012

Outline



1. Positive Systems
   - Definition
   - "Fathers"
   - Occurrence
   - Example
   - Publications
2. Monotone Systems
   - Definition
   - Early days
   - Publications

Positive System

A continuous linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, is called (internally) *positive* if and only if its state and output are nonnegative for every nonnegative input and every nonnegative initial state.

Theorem: Positivity [Luenberger, D. G., 1979]

A (cont.) linear system (A, B, C, D) is positive if and only if A is a Metzler-matrix and $B, C, D \geqq 0$.

"Fathers" of positive systems: Perron & Frobenius

Key result: Perron-Frobenius Theorem

(1849 - 1917)

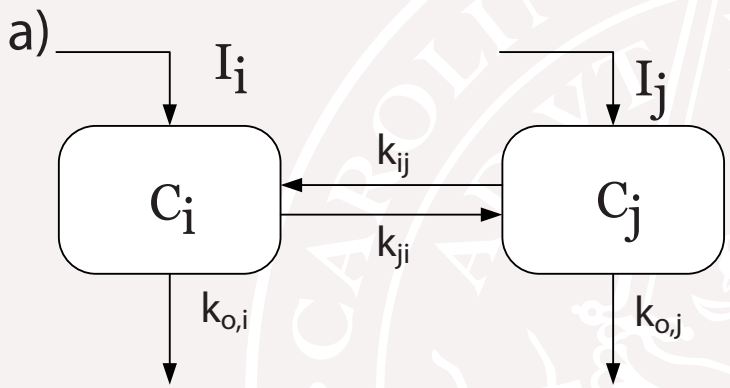
(1880 - 1975)

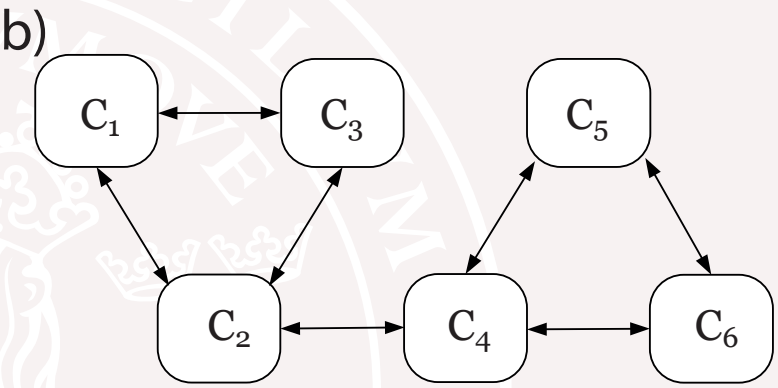
Occurrence of Positive systems

"[...]the positivity property just defined, is always nothing but the immediate consequence of the nature of the phenomenon we are dealing with. A huge number of examples are just before our eyes." [Farina, L., 2002]

- Network flows: traffic, transport, etc.
- Social science: population models
- Biology/Medicine: nitrade models, proteins, etc.
- Economy: stochastic models, markov jump systems, etc.
- Discretization of PDEs: heat equation

Example: Compartmental Network





$$\dot{x}_i(t) = -k_{o,i}x_i(t) + \sum_{j\neq i}^{n}[k_{ij}x_j(t) - k_{ji}x_i(t)] + \underbrace{\sum_{j=1}^{m} b_{ij}u_j(t)}_{I_i :=}$$

Publications: till 1999

Scopus: $\sim$ 70 publications mentioning positive systems.

Important ones:

- Introduction to Dynamic Systems: Theory, Models & Applications. (Luenberger 1979, Wiley)
- Reachability, observability and realizability of continuous-time positive systems. (Ohta 1984, SIAM)
- Nonnegative Matrices in Dynamical Systems (Berman 1989, Wiley)
- Robust stability of positive differentiable linear systems (Son, Hinrichsen 1995, CDC)

However, the term 'positive system' was and is still not commonly used:

- Lyapunov Functions for Diagonally Dominant Systems. (Willems 1976, Automatica)

Publications: 2000 - today

Scopus: $\sim$ 300 publications mentioning positive systems.

Important ones:

- Positive Linear Systems (Farina 2000, Wiley)
- Stabilization of positive linear systems (De Leenheer 2001, Systems & Control Letters)
- Stability of continuous-time distributed consensus algorithms (Moreau 2008, CDC)

In Europe most of the research in Italy and Belgium, but also some in Lund:

- Distributed control of positive systems (Rantzer 2011, CDC)
- Some result on model reduction of positive systems (Aivar and myself 2012)

But much theory hidden in the application, i.a.

- Love dynamics: The case of linear couples (Rinaldi 1998, Applied Mathematics and Computations)

Still missing

Difficult to solve and still missing:

- Transfer of the SISO-theory to MIMO.
- Adequate realization algorithms.

So far some attempts, however under highly conservative restrictions - pretty messy theory!

Monotone System

Let $\phi : X \subset V \to V$, where V is a real Banach space with an (partial) ordering $x \geqq y$ or a strongly ordering $x \gg y$.

A dynamical system, with solution flow ϕ, is called **monotone** if $\phi^t x \geqq \phi^t y$ for $t \geqq 0$ and $x \geqq y$ and **strongly monotone** if $\phi^t x \gg \phi^t y$ for $t > 0$ and $x \gg y$.

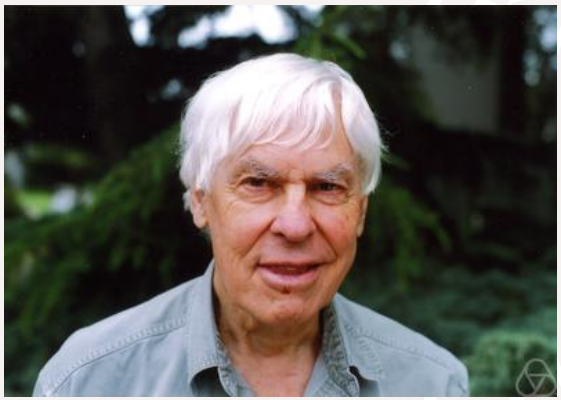
Proto-type: Cooperative system, which is the solution flow to a vector field F such that

$$\frac{\partial F_i}{\partial x_j} \geq 0 \text{ for } i \neq j.$$

If x_i denotes the population of a species i, then cooperative means, that an increase of x_i causes an increase in x_j.

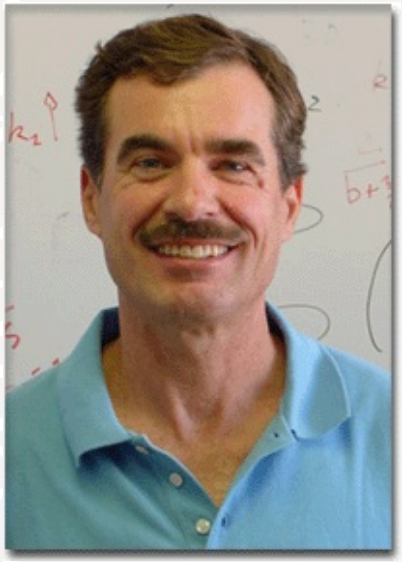
Early days: Hirsch, Smith & Smale

Key result: Convergence almost everywhere for strongly ordered systems (Hirsch 1981)



(Born 1930)

(Born 1933)



Publications: till 1999

Scopus: $\sim$ 230 publications mentioning monotone and cooperative systems.

Among many convergence results:

- Cooperative systems of differential equations with concave nonlinearities (Smith 1985)
- Stability and convergence in strongly monotone dynamical systems (Hirsch 1988)

Publications: 2000 - today

Scopus: $\sim$ 1600 publications mentioning monotone and cooperative systems.

Important ones:

- Monotone control systems (Angeli, Sontag 2003, IEEE TAC)
- Monotone Dynamical Systems - Chapter 4, Handbook of Differential Equations (Hirsch, Smith 2005)

Nowadays most attention on: Communication, Coordination and Biology.

- IFAC2005: $\sim$ 30 contributions (5 on positive systems)
- IFAC2008: $\sim$ 30 contributions (3 on positive systems)
- IFAC2011: $\sim$ 40 contributions (4 on positive systems)

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